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Minimal-Disturbance Measurement as a Specification in von Neumann's Quantal Theory of Measurement

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Abstract

In von Neumann's theory an incomplete observable A is measured by measuring any complete observable B whose function A is. This procedure is narrowed down in this paper by the additional requirement of preservation of the sharp value of any observable compatible with A. The requirement is shown to be equivalent to the unique change of state: $\rho \to (\mathrm{tr}\, \rho P_n)^{-1} P_n \rho P_n$ (P_n is the eigenprojector of A corresponding to the obtained eigenvalue a_n, ρ is the statistical operator of the initial state, and by assumption $\mathrm{tr}\, \rho P_n > 0$). This characterises the minimal-disturbance measurement. A necessary and sufficient condition is derived for the selection of the above observable B so that its measurement implies the minimal-disturbance measurement of A. For arbitrary ρ and A, there exists a B satisfying the condition. Hence, this constitutes a reasonable specification within von Neumann's theory, reducing the latter to the physically preferable minimal-disturbance measurement theory.

1. *Introduction*

If we have a quantal ensemble of physical systems which is described by a statistical operator ρ, we say, for simplicity, that we deal with a system in the state ρ. A system has a sharp real value d of an observable D in the state ρ if

$$\langle (D-d)^2 \rangle = \mathrm{tr}\, \rho (D-d)^2 = 0 \tag{1.1}$$

('tr' denoting the trace).

Lemma. A physical system has the sharp real value d of the observable D in the state ρ if for some decomposition of the latter into pure states $\rho = \Sigma_k w_k |\alpha_k\rangle\langle\alpha_k|$ ($w_k > 0$, $\forall\, k$; $\Sigma_k w_k = 1$; the number of terms finite or infinite), the eigenvalue equations

$$D|\alpha_k\rangle = d|\alpha_k\rangle, \qquad \forall\, k \tag{1.2}$$

are satisfied, and only if (1.2) are valid for every decomposition of ρ into pure states.

Proof. The sufficiency of equations (1.2) for the system to have the sharp value d of D in ρ is immediately seen. If $\rho = \Sigma_k w_k |\alpha_k\rangle\langle\alpha_k|$ is an arbitrary decomposition of ρ into pure states, then equation (1.1) implies

$$0 = \sum_k w_k \langle\alpha_k|(D-d)^2|\alpha_k\rangle = \sum_k w_k \|(D-d)|\alpha_k\rangle\|^2$$

(where '$\|\ldots\|$' denotes the norm), which has equations (1.2) as its immediate consequence. Q.E.D.

Remark 1. The sharp real value d of D in ρ necessarily equals the expectation value: $d = \langle D\rangle = \text{tr}\,\rho D$, and the measurement of D in ρ produces the result d with certainty.

According to von Neumann (1955, p. 220), in order to be able to perform an exact measurement of an observable A, this has to have a purely discrete spectrum. Let the unique spectral form† of A be

$$A = \sum_n a_n P_n \tag{1.3}$$

where a_n are the distinct eigenvalues of A, and P_n are the corresponding eigenprojectors. The degeneracy of a_n is $\text{tr}\,P_n \geqslant 1$. Let ρ be the initial state of the system, and let N be the sufficiently large number of physical systems in the quantal ensemble described by ρ. As is well known, the standard *concept of measurement* (von Neumann, 1955) is based on the following two requirements:

Requirement 1. Having measured A, one may group the N systems into non-overlapping classes enumerated by n and containing N_n systems each ($\Sigma_n N_n = N$), so that if $N_n > 0$, the N_n systems form a quantal sub-ensemble in which A has the sharp value a_n.

Requirement 2. For each n, one has with sufficient accuracy $N_n/N \simeq \text{tr}\,\rho P_n$ (the probability to obtain a_n measuring A in ρ).

If $\text{tr}\,P_n = 1$, $\forall\, n$ in equation (1.3), or, as it is said, A is a *complete* observable, then for each a_n there exists a (up to a phase factor) unique normalised eigenstate $|\varphi_n\rangle$, so that $A = \Sigma_n a_n |\varphi_n\rangle\langle\varphi_n|$ (all a_n distinct). Utilising the standard connection between the decomposition of an ensemble and that of the corresponding statistical operator ($N = \Sigma_n N_n \leftrightarrow \rho = \Sigma_n w_n \rho_n$, $w_n \simeq N_n/N$), it is easy to see that Requirements 1 and 2 are in this case equivalent to the following unique form of the statistical operator describing the entire ensemble after the measurement:

$$\sum_n \langle\varphi_n|\rho|\varphi_n\rangle |\varphi_n\rangle\langle\varphi_n| \tag{1.4}$$

† We call 'unique' that spectral form of an observable A with a purely discrete spectrum in which no repetition of eigenvalue occurs and the projectors add up to 1.

If A is an *incomplete* observable, i.e., if it is not in itself a complete set of commuting observables, like in the case just mentioned, then von Neumann (1955, p. 348) introduces a complete observable B:

$$B \equiv \sum_n \sum_{i=1}^{I_n} b_{n,i} |\varphi_{n,i}\rangle\langle\varphi_{n,i}| \tag{1.5}$$

all $b_{n,i}$ distinct, so that

$$\sum_{i=1}^{I_n} |\varphi_{n,i}\rangle\langle\varphi_{n,i}| = P_n, \qquad \forall\, n \tag{1.6}$$

($I_n = \text{tr}\, P_n$ is an integer or ∞).† Relation (1.6) is tantamount to

$$A = f(B) \tag{1.7}$$

where '$f(\ldots)$' designates an operator function.

The application of expression (1.4) to the case of measuring B, after replacing n by n, i, gives:

$$\rho'(B) = \sum_n (\text{tr}\, \rho P_n)\rho'_n(B) \tag{1.8}$$

with

$$\rho'_n(B) = \sum_{i=1}^{I_n} (\langle\varphi_{n,i}|\rho|\varphi_{n,i}\rangle/\text{tr}\, \rho P_n)|\varphi_{n,i}\rangle\langle\varphi_{n,i}| \tag{1.9}$$

Owing to (1.6),

$$\text{tr}\, \rho P_n = \sum_{i=1}^{I_n} \langle\varphi_{n,i}|\rho|\varphi_{n,i}\rangle \tag{1.10}$$

In equation (1.9) by assumption $\text{tr}\, \rho P_n > 0$, otherwise $\rho'_n(B)$ is not defined.

Von Neumann (1955) calls the process giving rise to the transition $\rho \rightarrow \rho'(B)$ 'process one'. Using a more modern terminology, we refer to this as to the non-selective measurement of A in ρ; and for the transition $\rho \rightarrow \rho'_n(B)$ we say that it is due to the selective measurement of A in ρ corresponding to the eigenvalue a_n.

One concludes from equations (1.8), (1.9) and (1.6) that $\rho'(B)$ satisfies the above two requirements for the measurement of A because $\text{tr}\, \rho P_n$ stands for N_n/N, and in each $\rho'_n(B)$ the system has the sharp value a_n of A (cf. the lemma). Thus, the measurement of A can be performed via measuring *any* B (satisfying equations (1.5) and (1.6)), though neither the selection of B nor the ensuing changed state $\rho'(B)$ is unique, in general. So far the quantal theory of measurement of von Neumann.

† For simplicity we assume that the Hilbert space of the system is finite or countably-infinite dimensional.

2. *Minimal-Disturbance Measurement*

The question arises: what is the role played by an incomplete observable A in quantum mechanics? The answer is well known: one forms a sequence of compatible observables $D, A, \ldots$ to achieve a complete set. This enables one to arrive at a pure state whenever one has a sequence of corresponding eigenvalues, i.e., sharp values: $d, a_n, \ldots$, which are achieved by performing a *succession* of measurements (of D, A, etc.). For this accumulation of sharp values, it is indispensable that each measurement should preserve the ones already acquired by the system in the previous measurements of compatible observables. Therefore, one has to complete the concept of measurement of a general observable A (with a purely discrete spectrum) by adding the following requirement to the above two.

Requirement 3. If a physical system in a state ρ has the *sharp* real value d of an observable D, and if this observable is *compatible* with the measured observable A, then whichever result a_n of the measured observable A is obtained, the N_n systems having produced this result have to be in a state ρ_n' in which the sharp value d of D is *preserved.*

Theorem 1. A. If the measurement of an observable A—whose spectral form is given by equation (1.3)—satisfies Requirements 1 and 3, then an arbitrary state ρ is changed into the state

$$\rho_n' = (\mathrm{tr}\, \rho P_n)^{-1} P_n \rho P_n \tag{2.1}$$

in the selective measurement of A corresponding to the eigenvalue a_n whenever $\mathrm{tr}\, \rho P_n > 0$.

B. If $\mathrm{tr}\, \rho P_n > 0$ implies equation (2.1) for the change of state, then Requirement 3 is valid.

Before we prove the theorem, let us point out that, as a consequence of equation (2.1) and Requirement 2, the non-selective measurement of A in ρ results in the state

$$\rho' = \sum_n (\mathrm{tr}\, \rho P_n)\rho_n' = \sum_n P_n \rho P_n \tag{2.2}$$

Further, if the initial state is pure: $\rho = |\psi\rangle\langle\psi|$ ($|\psi\rangle$ being a normalised state vector), then equation (2.1) boils down to

$$|\psi_n\rangle = \langle\psi|P_n|\psi\rangle^{-1/2} P_n|\psi\rangle \tag{2.3}$$

(assuming $\langle\psi|P_n|\psi\rangle > 0$), where $|\psi_n\rangle\langle\psi_n|$ plays the role of ρ'.

Proof. A. We perform the proof in two steps. First we demonstrate the special case of equation (2.3), then we generalise to equation (2.1).

Let us assume that $\rho = |\psi\rangle\langle\psi|$. Renumerating by m the non-zero ones of the projections $P_n|\psi\rangle$, $n = 1, 2, \ldots$, we define

$$|\psi_m\rangle \equiv \langle\psi|P_m|\psi\rangle^{-1/2} P_m|\psi\rangle, \qquad m = 1, 2, \ldots \tag{2.4}$$

and

$$D \equiv \sum_m |\psi_m\rangle\langle\psi_m| \tag{2.5}$$

Since one can write $|\psi\rangle = \Sigma_m \langle\psi|P_m|\psi\rangle^{1/2}|\psi_m\rangle$, obviously

$$D|\psi\rangle = |\psi\rangle \tag{2.6}$$

i.e., in the state $\rho = |\psi\rangle\langle\psi|$ the observable D has the sharp value 1. On the other hand, equation (2.5) implies $[D, P_n]_- = 0$, $\forall\, n$, and, owing to equation (1.3), this leads to

$$[D, A]_- = 0 \tag{2.7}$$

i.e., D is compatible with A. Then, by Requirement 3, D has the sharp value 1 also in every ρ_n' with n such that $\langle\psi|P_n|\psi\rangle > 0$. By Requirement 1, in ρ_n' the observable A has the sharp value a_n. Let $\rho_n' = \Sigma_k w_k |\alpha_k\rangle\langle\alpha_k|$ be an arbitrary decomposition of ρ_n' into pure states. There exists at least one such decomposition, viz., the spectral form of ρ_n', because each statistical operator has a purely discrete spectrum (von Neumann, 1955, p. 329).

According to the lemma, the mentioned sharp values in ρ_n' imply $D|\alpha_k\rangle = |\alpha_k\rangle$, $\forall\, k$, and $A|\alpha_k\rangle = a_n|\alpha_k\rangle$, $\forall\, k$. The latter equations are equivalent to $P_n|\alpha_k\rangle = |\alpha_k\rangle$, $\forall\, k$. As a consequence,

$$P_n D|\alpha_k\rangle = |\alpha_k\rangle, \qquad \forall\, k \tag{2.8}$$

From equation (2.5) one can see that

$$P_n D = |\psi_n\rangle\langle\psi_n|, \qquad \text{if } \langle\psi|P_n|\psi\rangle > 0 \tag{2.9}$$

Hence, taking once more resort to the lemma, we conclude from equation (2.8) that in ρ_n' the observable $|\psi_n\rangle\langle\psi_n|$ has the sharp value 1. This is possible only if $\rho_n' = |\psi_n\rangle\langle\psi_n|$, where $|\psi_n\rangle$ is defined by equation (2.3).

In operator form equation (2.3) can be written

$$\rho_n' = \langle\psi|P_n|\psi\rangle^{-1} P_n|\psi\rangle\langle\psi|P_n \tag{2.10}$$

for $\rho = |\psi\rangle\langle\psi|$.

Now, let the initial state ρ be, in the terminology of D'Espagnat (1971), a proper mixture, in which there are admixed, e.g., K pure states:†

$$\rho = \sum_{k=1}^{K} w_k |\psi_k\rangle\langle\psi_k| \tag{2.11}$$

($w_k > 0$, $\forall\, k$; $\Sigma_k w_k = 1$). Let ρ describe a quantal ensemble of N physical systems, and $|\psi_k\rangle\langle\psi_k|$ that of N_k of them ($N = \Sigma_{k=1}^K N_k$, N_k sufficiently large, $\forall\, k$; $N_k/N \simeq w_k$). The selective measurement of A corresponding to a_n

† K is necessarily finite because one cannot prepare a mixture containing an infinite number of pure states. One may admit formally $K = \infty$ in proper mixtures extending to this case by stipulation the formulae for finite K.

in the initial state ρ results in, say, N_n systems ($N = \Sigma_n N_n$), of which $N_n^{(k)}$ stem from the N_k initial ones:

$$N_n = \sum_{k=1}^{K} N_n^{(k)} \tag{2.12}$$

If a_n is detectable in ρ, the N_n systems are described by the so far unknown statistical operator ρ_n', and the $N_n^{(k)}$ systems by

$$\rho_n'^{(k)} \equiv \langle \psi_k | P_n | \psi_k \rangle^{-1} P_n | \psi_k \rangle \langle \psi_k | P_n \tag{2.13}$$

(cf. equation (2.10) with $|\psi\rangle = |\psi_k\rangle$), unless $N_n^{(k)} = 0$ (i.e., $\langle \psi_k | P_n | \psi_k \rangle = 0$). All we need are the weights w_k' in the decomposition of the statistical operator ρ_n':

$$\rho_n' = \sum_{k=1}^{K} w_k' \rho_n'^{(k)} \tag{2.14}$$

which corresponds to equation (2.12) (in case $\rho_n'^{(k)}$ is not defined, w_k' has to be zero). Evidently, $w_k' \simeq N_n^{(k)}/N_n$ (a_n is assumed detectable in ρ, i.e., $N_n > 0$). Since

$$N_n^{(k)}/N_n = (N_n^{(k)}/N_k)(N_k/N)(N/N_n) \tag{2.15}$$

and $N_n^{(k)}/N_k \simeq \langle \psi_k | P_n | \psi_k \rangle, N_k/N \simeq w_k, N/N_n \simeq (\mathrm{tr}\, \rho P_n)^{-1}$, we have

$$w_k' = \langle \psi_k | P_n | \psi_k \rangle w_k (\mathrm{tr}\, \rho P_n)^{-1} \tag{2.16}$$

If $N_n^{(k)} = 0$ ($\rho_n'^{(k)}$ undefined), then $\langle \psi_k | P_n | \psi_k \rangle = 0$, and via equation (2.16) $w_k' = 0$ as required. The detectability of a_n in ρ means $\mathrm{tr}\, \rho P_n > 0$, so that equation (2.16) is consistent. Finally, we obtain from equations (2.14), (2.16), (2.13) and (2.11):

$$\begin{aligned}\rho_n' &= \sum_k \langle \psi_k | P_n | \psi_k \rangle w_k (\mathrm{tr}\, \rho P_n)^{-1} \langle \psi_k | P_n | \psi_k \rangle^{-1} P_n | \psi_k \rangle \langle \psi_k | P_n \\ &= (\mathrm{tr}\, \rho P_n)^{-1} P_n \rho P_n\end{aligned}$$

which is the claimed expression (2.1).

Now let the initial state be, in the terminology of D'Espagnat (1971), an improper mixture, i.e., let ρ be actually the reduced statistical operator ρ_1 of a composite system (consisting of subsystems 1 and 2) in a pure state $|\phi_{12}\rangle$:

$$\rho_1 = \mathrm{tr}_2 |\phi_{12}\rangle\langle\phi_{12}| \tag{2.17}$$

('tr_2' denoting the partial trace over all the coordinates of the second subsystem). Then the observable A is actually a first-subsystem observable A_1, i.e., one measures $A_1 \otimes 1_2$ in the state $|\phi_{12}\rangle$ to obtain

$$\rho_{12}'^{(n)} = \langle \phi_{12} | P_1^{(n)} | \phi_{12} \rangle^{-1} P_1^{(n)} | \phi_{12} \rangle \langle \phi_{12} | P_1^{(n)} \tag{2.18}$$

in the selective measurement corresponding to a_n (cf. equation (2.10), and note that in equation (1.3) P_n is replaced by $P_1^{(n)}$). What equation (2.18) implies for the first subsystem is derived by the partial trace over the coordinates of the second subsystem. One obtains

$$\rho_1'^{(n)} = (\mathrm{tr}_1 P_1^{(n)} \rho_1)^{-1} P_1^{(n)} \rho_1 P_1^{(n)} \tag{2.19}$$

Namely, the probability to obtain a_n in $|\phi_{12}\rangle$, i.e., $\langle\phi_{12}|P_1^{(n)}|\phi_{12}\rangle$ equals $\mathrm{tr}_1 P_1^{(n)} \rho_1$, as well known; and first-subsystem observables, such as $P_1^{(n)}$, can be taken outside the partial trace over the second subsystem. Equation (2.19) again equals (2.1).

Finally, let us consider the case of a proper mixture of pure composite states $\rho_{12} = \Sigma_{k=1}^K w_k |\phi_{12}^{(k)}\rangle\langle\phi_{12}^{(k)}|$. For the first subsystem this implies the reduced statistical operator $\rho_1 = \Sigma_{k=1}^K w_k \rho_1^{(k)}$, where $\rho_1^{(k)} = \mathrm{tr}_2 |\phi_{12}^{(k)}\rangle\langle\phi_{12}^{(k)}|$, which is a proper mixture of improper mixtures. Now, if we measure the first-subsystem observable A_1 and obtain a_n, the effect on ρ_1 equals that on ρ_{12} with subsequent reduction to the first subsystem:

$$\rho_{12} \to (\mathrm{tr}_{12} P_1^{(n)} \rho_{12})^{-1} P_1^{(n)} \rho_{12} P_1^{(n)} \to (\mathrm{tr}_1 P_1^{(n)} \rho_1)^{-1} P_1^{(n)} \rho_1 P_1^{(n)}$$

as asserted in (2.1). This completes the derivation of (2.1) in the general case.

B. To prove the second claim in Theorem 1, we assume $\mathrm{tr}\,\rho(D-d)^2 = 0$, d real. Since $\Sigma_n P_n = 1$ (cf. equation (1.3)), one has $\Sigma_{nn'}\,\mathrm{tr}\,P_n \rho P_{n'}(D-d)^2 = 0$. Assuming further that $[D, P_{n'}]_- = 0, \forall n'$ (equivalent to $[D, A]_- = 0$), we have $\Sigma_{nn'}\,\mathrm{tr}\,P_n P_{n'} \rho (D-d)^2 = 0$. As $P_n P_{n'} = \delta_{nn'} P_n$, one further obtains $\Sigma_n\,\mathrm{tr}\,P_n \rho (D-d)^2 P_n = 0$. Owing to $\mathrm{tr}\,P_n \rho (D-d)^2 P_n = \mathrm{tr}\,(D-d) P_n \rho P_n (D-d) \geqslant 0$ (because $\langle u|(D-d)P_n\rho P_n(D-d)|u\rangle = \|\rho^{1/2} P_n (D-d)|u\rangle\|^2 \geqslant 0$, $\forall\,|u\rangle$), one finally has $\mathrm{tr}\,P_n \rho P_n (D-d)^2 = 0$, $\forall\,n$, leading to

$$\mathrm{tr}\,\rho_n'(D-d)^2 = 0 \tag{2.20}$$

whenever $\mathrm{tr}\,\rho P_n > 0$ (ρ_n' defined by (2.1)). In conjunction with the assumptions made, equation (2.20) is obviously equivalent to Requirement 3. Q.E.D.

Measurement involving the change of state given by equation (2.1) has come to be called *minimal-disturbance measurement.* Reasons for this term (besides Requirement 3) will be apparent in Section 4.

3. *The Specification Required in Von Neumann's Theory*

Before we come back to von Neumann's theory, outlined in the Introduction, let us note the fact that the statistical operators ρ_n' and ρ', determined by equations (2.1) and (2.2) respectively, commute with each eigenprojector $P_{n'}$ of A, and hence also with A itself. In the state ρ_n' the observable A has the sharp value a_n, therefore, making use of the lemma, one may conclude that every eigenvector of ρ_n' corresponding to a positive eigenvalue is also an eigenvector of A, corresponding to a_n. Utilising this explicitly, if $\mathrm{tr}\,\rho P_n > 0$, ρ_n' can be written in the spectral form

$$\rho_n' = \sum_{s=1}^{S_n} r_{n,s}' Q_{n,s} \tag{3.1}$$

where all eigenvalues $r'_{n,s}$ are distinct, and the orthogonal projectors $Q_{n,s}$ add up to P_n:

$$\sum_{s=1}^{S_n} Q_{n,s} = P_n \tag{3.2}$$

($S_n \leqslant \text{tr}\, P_n$ is an integer or ∞).

One should note that for $r'_{n,s} > 0$, $Q_{n,s}$ is the corresponding eigenprojector of ρ'_n; if $r'_{n,s_0} = 0$ (which may appear), then $Q_{n,s_0} \equiv P_n - \Sigma_{s \neq s_0} Q_{n,s}$, and $(Q_{n,s_0} + 1 - P_n)$ is the eigenprojector of ρ'_n corresponding to the eigenvalue zero.

Equations (2.2) and (3.1) entail for ρ' the spectral form

$$\rho' = \sum_n \text{tr}\, \rho P_n \sum_{s=1}^{S_n} r'_{n,s} Q_{n,s} \tag{3.3}$$

If n is such that $\text{tr}\, \rho P_n = 0$, then we put $S_n = 1$, $r'_{n,1} = 0$, and $Q_{n,1} = P_n$. In this way, equation (3.2) is valid for all n. Further, the decomposition of the identity

$$\sum_n \sum_{s=1}^{S_n} Q_{n,s} = 1 \tag{3.4}$$

is the unique common eigen-decomposition corresponding to A and ρ' (distinct $Q_{n,s}$ correspond to distinct pairs of eigenvalues a_n, $r'_{n,s}$ of A, ρ').

Now we can raise and answer the question if, for an arbitrary initial state ρ and an arbitrary observable A, there exists a complete observable B, restricted by (1.6), whose measurement would lead via equations (1.8) and (1.9) to the minimal-disturbance measurement expression of ρ' (given by (2.2)). The answer is affirmative.

Theorem 2. Equation (1.8) takes the form (2.2) if and only if the complete observable B, whose non-selective measurement in ρ gives (1.8), has the spectral form

$$B = \sum_n \sum_{s=1}^{S_n} \sum_{t=1}^{T_{n,s}} b_{n,s,t} |\varphi_{n,s,t}\rangle\langle\varphi_{n,s,t}| \tag{3.5}$$

(all $b_{n,s,t}$ distinct), so that

$$\sum_{t=1}^{T_{n,s}} |\varphi_{n,s,t}\rangle\langle\varphi_{n,s,t}| = Q_{n,s}, \qquad \forall\, s, \forall n \tag{3.6}$$

($T_{n,s} = \text{tr}\, Q_{n,s}$). For any ρ and any A with a purely discrete spectrum, there exists a B of this form.

Proof. Sufficiency. For any B given by (1.5) with (1.6), one can rewrite (1.8) in the form

$$\rho'(B) = \sum_n \sum_{i=1}^{I_n} \langle \varphi_{n,i} | \rho' | \varphi_{n,i} \rangle | \varphi_{n,i} \rangle \langle \varphi_{n,i} | \tag{3.7}$$

with ρ' defined by (2.2). This is due to the fact that (1.6) implies

$$P_n | \varphi_{n',i} \rangle = \delta_{nn'} | \varphi_{n,i} \rangle, \qquad \forall\, n, n', i \tag{3.8}$$

Let B satisfy (3.5) with (3.6). (Note that the latter implies via (3.2) the restriction (1.6).) Then we can replace i by s, t in (3.7):

$$\rho'(B) = \sum_n \sum_{s=1}^{S_n} \sum_{t=1}^{T_{n,s}} \langle \varphi_{n,s,t} | \rho' | \varphi_{n,s,t} \rangle | \varphi_{n,s,t} \rangle \langle \varphi_{n,s,t} | \tag{3.9}$$

Equations (3.6) and (3.3) tell us that the $|\varphi_{n,s,t}\rangle$ are eigenvectors of ρ', so that the right-hand side of (3.9) equals ρ', as claimed.

Necessity. Let us assume that $\rho' = \rho'(B)$, and that B is given by equation (1.5) with (1.6). Replacing here ρ' from (2.2), the P_n from (1.6), and $\rho'(B)$ from (1.8) with (1.9), we can write

$$\sum_n \sum_{i'=1}^{I_n} \sum_{j'=1}^{I_n} | \varphi_{n,i'} \rangle \langle \varphi_{n,i'} | \rho | \varphi_{n,j'} \rangle \langle \varphi_{n,j'} |$$

$$= \sum_n \sum_{i'=1}^{I_n} \langle \varphi_{n,i'} | \rho | \varphi_{n,i'} \rangle | \varphi_{n,i'} \rangle \langle \varphi_{n,i'} |$$

Taking the matrix element $\langle \varphi_{n,i} | \ldots | \varphi_{n,j} \rangle$ of both sides, one has $\langle \varphi_{n,i} | \rho | \varphi_{n,j} \rangle = 0$, if $i \neq j$. Owing to (3.8) and (2.2), one can write this as $\langle \varphi_{n,i} | \rho' | \varphi_{n,j} \rangle = 0$, $i \neq j$, or, due to the quasidiagonality of ρ' in n (cf. (2.2)), one finally has

$$\langle \varphi_{n,i} | \rho' | \varphi_{n',j} \rangle = 0 \qquad \text{if } n \neq n' \text{ or } i \neq j \tag{3.10}$$

Equation (3.10) means that $\{|\varphi_{n,i}\rangle, i = 1, 2, \ldots, I_n, n = 1, 2, \ldots\}$ is an eigenbasis of ρ'. On the other hand, (1.6) and (1.3) mean that this same basis is an eigenbasis of A, i.e., it is a common eigenbasis of A and ρ'. Hence, each basis element belongs to the range of some $Q_{n,s}$ from (3.4), and we can relabel the basis elements with n, s, t to obtain the spectral form (3.5) with (3.6) valid for B as asserted in Theorem 2.

That for every ρ and A there exists a B satisfying (3.5) with (3.6) follows from the fact that the commuting projectors $Q_{n,s}$, $s = 1, 2, \ldots, S_n$, $n = 1, 2, \ldots$ certainly have a simultaneous eigenbasis $\{|\varphi_{n,s,t}\rangle, t = 1, 2, \ldots, T_{n,s}, s = 1, 2, \ldots, S_n, n = 1, 2, \ldots\}$ (and the $b_{n,s,t}$ are arbitrary distinct real numbers). Q.E.D.

The condition in Theorem 2 is a *specification* in von Neumann's theory of measurement in the sense that, in general, it eliminates some 'wrong' B's (satisfying (1.5) and (1.6) but not (3.5) and (3.6)), leaving a non-empty set of 'good' B's, the measurement of each of which performs the measurement of A in accordance with all three requirements.

Remark 2. The necessary and sufficient condition on the complete observable B formulated in Theorem 2 can be put in the form

$$A = f(B), \qquad \rho' = g(B) \tag{3.11}$$

where '$g(\ldots)$' denotes an operator function just like '$f(\ldots)$' (cf. (1.7)).

Remark 3. The selection of the 'good' B's depends on the initial state ρ (through ρ'). Nevertheless, the change of state $\rho \to \rho' = \Sigma_n P_n \rho P_n$ in minimal-disturbance measurement 'knows nothing' about the B by means of which the measurement was performed. The measured observable A associates (via its eigenprojectors P_n) with every ρ a unique ρ'. It is natural to stipulate with Lüders (1951) that to this operator map $\rho \to \rho'$ there corresponds a unique laboratory procedure (depending only on A not on ρ), which brings about the *incomplete measurement*† of A in any initial state. For every given ρ this measurement is then equivalent to the complete measurement of any 'good' B.

For a *pure* initial state $|\psi\rangle$, ρ' takes the form

$$\rho' = \sum_n \langle\psi|P_n|\psi\rangle |\psi_n\rangle\langle\psi_n| \tag{3.12}$$

where $|\psi_n\rangle$ is given by (2.3). In this case the condition on the 'good' B's is particularly simple:

Corollary. For a pure initial state $\rho = |\psi\rangle\langle\psi|$, one has $\rho'(B) = \rho'$ if and only if one restricts the choice of the complete observable B (specified by (1.5) and (1.6)) by the following additional requirement: For every value of n such that $\langle\psi|P_n|\psi\rangle > 0$, there exists a value of i, say i_0, for which

$$|\varphi_{n,i_0}\rangle = e^{i\lambda_n}|\psi_n\rangle \tag{3.13}$$

(cf. (2.3)), and the phase factor is arbitrary.

4. *Relation to the Literature*

As explained in the Introduction, von Neumann's theory reduces the measurement of any observable to a complete measurement. The correctness of this procedure was challenged by Lüders (1951), who postulated expression (2.2) in conjunction with (1.3) for incomplete measurement.

† 'Incomplete measurement' actually means measurement of an incomplete observable. But, rejecting the 'wrong' B's, because their measurement violates Requirement 3 for A, one may use this term as a synonym for minimal-disturbance measurement.

Wigner, who was perhaps unaware of Lüders' work, and who equally felt the need to introduce direct incomplete measurement, was said to have arrived at the $|\psi\rangle \to |\psi_n\rangle$ transition via an argument of maximal overlap, which he stated to be 'morally best' (Goldberger & Watson, 1964).

Bell & Nauenberg (1966) accepted Wigner's ethical term for his essentially variational concept, and spoke of a moral measurement, causing D'Espagnat (1971) to follow them. They pointed out that the $|\psi\rangle \to |\psi_n\rangle$ transition could be obtained from the requirement: If A and D commute (compatible observables) and a_n and d_m are their respective eigenvalues, the probability to obtain a_n, d_m should be the same irrespectively whether one performs a joint incomplete measurement of A and D in $|\psi\rangle$, or one does their incomplete measurements in immediate succession.

In an earlier paper, this author (Herbut, 1969) presented two derivations of formula (2.2) unaware of the mentioned references except of Lüders' (1951) pioneering work. The first derivation was along variational lines. It was based on the fact that statistical operators ρ belong to the Hilbert space of all linear Hilbert–Schmidt operators acting in the Hilbert space of state vectors,† and thus there exists a natural concept of distance between two statistical operators. The change $\rho \to \rho'$ was required to minimise the ρ, ρ'-distance under Requirement 1 as a restriction. In accordance with these ideas, the incomplete measurement of A was called its minimal measurement.

The second derivation of Lüders' formula (2.2) (Herbut, 1969) required the expectation values before and after the measurement of A, i.e., tr ρD and tr $\rho' D$, to coincide if D is compatible with A. It is straightforward to convince oneself that this requires in terms of the entire ensemble after the measurement and the expectation values the same thing as Bell & Nauenberg (1966) required in the language of the resulting subensemble corresponding to a_n (i.e., ρ'_n), and the probabilities. Both these derivations are essentially based on the idea that measurement should not destroy information on compatible observables. The approach founded on this idea is an alternative to the variational one.

The derivation of (2.2) given in Theorem 1 in this paper belongs to the conceptual framework of the compatibility approach. It requires physically and logically the least, because the class of all observables D having a sharp value d in the initial state and being compatible with A, is a very restricted subclass of the class of all observables compatible with A. On the other hand, Requirement 3 has the same advantage as Requirement 1, both can be put in terms of properties of *individual* physical systems since sharp values in an ensemble can be interpreted as belonging to each element (each individual physical system) in the ensemble.

The main result of this paper is Theorem 2 showing that minimal-disturbance measurement theory is not a rival to von Neumann's theory, but rather a natural elaboration of the latter in the sense of Requirement 3 (leading via its equivalent (2.2) to the necessary and sufficient condition on 'good' B's).

† The space of the linear Hilbert–Schmidt operators is called superspace and is made extensive use of in George, C., Prigogine, I. and Rosenfeld, L. (1972). *Dan. Vid. Selsk. mat.-fys. Medd.* **38**, No. 12.

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References

Bell, J. S. and Nauenberg, M. (1966). *Preludes in Theoretical Physics.* North-Holland, Amsterdam.

D'Espagnat, B. (1971). *Conceptual Foundations of Quantum Mechanics.* W. A. Benjamin, California.

Goldberger, M. L. and Watson, K. M. (1964). *Physical Review,* **134,** B919.

Herbut, F. (1969). *Annals of Physics,* **55**, 271.

Lüders, G. (1951). *Annalen der Physik,* **8**, 322.

Von Neumann, J. (1955). *Mathematical Foundation of Quantum Mechanics.* Princeton University Press, Princeton.